Problem 17) Separation of variables: Let $T(r,\phi,t) = f(r)g(\phi)h(t)$. Substitution into the heat diffusion equation yields:

$$D\nabla^2 T = D\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}\right) = \frac{\partial T}{\partial t}$$

$$\rightarrow \quad D[f''(r)g(\phi)h(t) + r^{-1}f'(r)g(\phi)h(t) + r^{-2}f(r)g''(\phi)h(t)] = f(r)g(\phi)h'(t).$$

Dividing both sides of the above equation by $f(r)g(\phi)h(t)$, we will have

$$D\left[\frac{f''(r)}{f(r)} + r^{-1}\frac{f'(r)}{f(r)} + r^{-2}\frac{g''(\phi)}{g(\phi)}\right] = \frac{h'(t)}{h(t)}.$$

Now, the right-hand side of this equation is independent of r and ϕ, which must therefore be set equal to a constant. We choose the real, negative constant $-c$ for $h'(t)/h(t)$, to ensure that the solution, $h(t) = \exp(-ct)$, does *not* grow with time. Similarly, we choose the real, negative constant $-\alpha^2$ for $g''(\phi)/g(\phi)$, to ensure solutions of the form $g(\phi) = \exp(\pm i\alpha\phi)$, which become periodic (with the required period of $\Delta\phi = 2\pi$) when α is an integer, say, $\alpha = m$. Since the initial temperature above ambient at $t = 0$ is known to be in the form of $f(r)\cos\phi$, the only acceptable solution for $g(\phi)$ is $\cos\phi$ and, therefore, $m = \alpha = 1$. The remaining part of the equation is now written as follows:

$$D\left[\frac{f''(r)}{f(r)} + r^{-1}\frac{f'(r)}{f(r)} - r^{-2}\right] = -c \quad \rightarrow \quad r^2 f''(r) + rf'(r) + [(c/D)r^2 - 1]f(r) = 0.$$

This is the Bessel equation of order 1, whose two independent solutions are $J_1(\sqrt{c/D}\,r)$ and $Y_1(\sqrt{c/D}\,r)$. Since $Y_1(\cdot)$ diverges at the center of the disk, the only acceptable solution is going to be $J_1(\sqrt{c/D}\,r)$. Moreover, at the boundary $r = R$ of the disk, the temperature is fixed at the ambient temperature T_0, which forces the solution $J_1(\sqrt{c/D}\,r)$ to go to zero at this boundary. Denoting the n^{th} zero of $J_1(\rho)$ by ρ_{1n}, we will have $\sqrt{c/D}\,R = \rho_{1n}$, which yields $c = D\,\rho_{1n}^2/R^2$. The complete solution of the heat diffusion equation for the present problem is thus given by

$$T(r,\phi,t) = T_0 + \sum_{n=1}^{\infty} A_n J_1(\rho_{1n} r/R)\cos\phi \exp(-D\rho_{1n}^2 t/R^2).$$

To find the unknown coefficients A_n, we resort to the initial condition at $t = 0$, which requires that $f(r) = \sum_{n=1}^{\infty} A_n J_1(\rho_{1n} r/R)$ in the interval $0 \le r \le R$. The coefficients A_n are readily obtained using the orthogonality of the Bessel functions $J_1(\rho_{1n} r/R)$ for different values of n, that is,

$$\int_0^R rf(r)J_1(\rho_{1m} r/R)\,\mathrm{d}r = \sum_{n=1}^{\infty} A_n \int_0^R rJ_1(\rho_{1n} r/R)\,J_1(\rho_{1m} r/R)\,\mathrm{d}r = A_m \int_0^R rJ_1^2(\rho_{1m} r/R)\,\mathrm{d}r$$

$$\rightarrow \quad A_m = \int_0^R rf(r)J_1(\rho_{1m} r/R)\,\mathrm{d}r \Big/ \int_0^R rJ_1^2(\rho_{1m} r/R)\,\mathrm{d}r.$$
